Chair of Distributed Systems and Security
School of Computation, Information and Technology
Technical University of Munich

Computer Networking and IT Security (INHN0012)

Tutorial 2

Problem 1 Signal Analysis and Synthesis

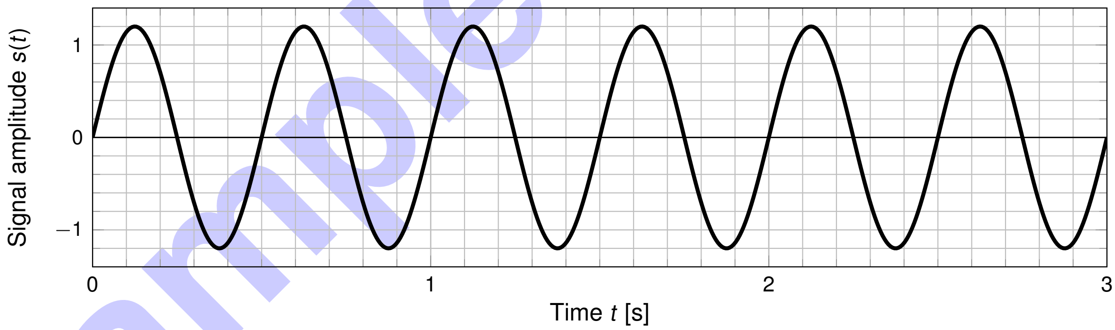
In general, signals can be represented either in the *time domain* or in the *frequency domain*. These two representations can be transformed into each other through Fourier analysis or synthesis. The given signals are periodic in the time domain; therefore, their frequency spectrum can be analyzed using the Fourier series.

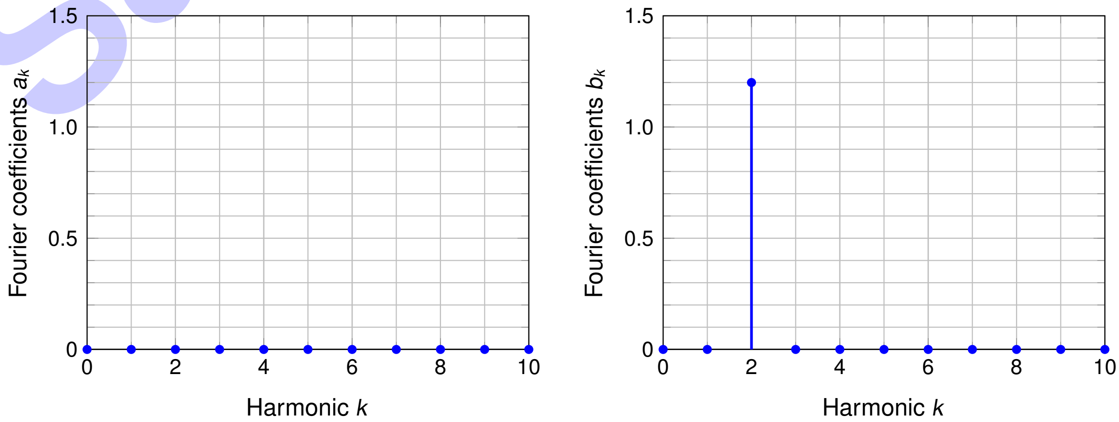
a) For what reason is the spectrum of a signal or its representation in the frequency domain of interest?

A periodic signal consists of the superposition of several harmonic oscillations. A non-ideal channel often exhibits certain filtering effects, such that different frequencies are attenuated differently or transmitted with varying quality. Filters can, in general, be applied and analyzed mathematically more easily in the frequency domain.
The superposition of multiple harmonic oscillations also appears in acoustics. A tone produced by an instrument usually consists of a fundamental oscillation / a fundamental tone, as well as several overtones. These can be identified through frequency analysis.
Further applications of Fourier analysis include optics, digital image and audio processing.

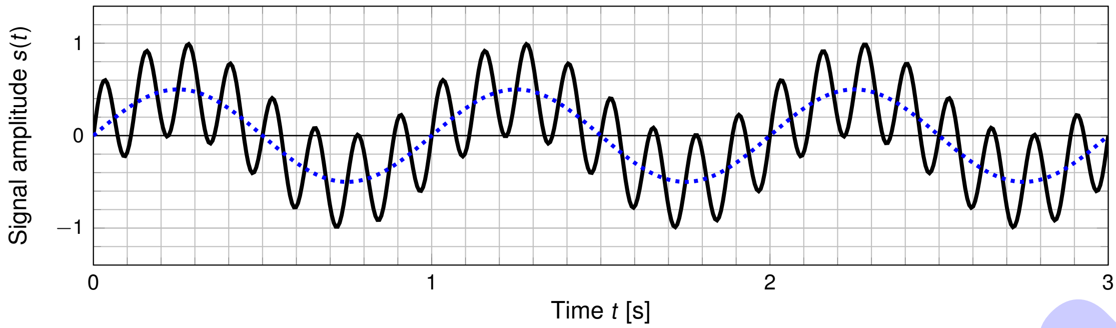
Analysis

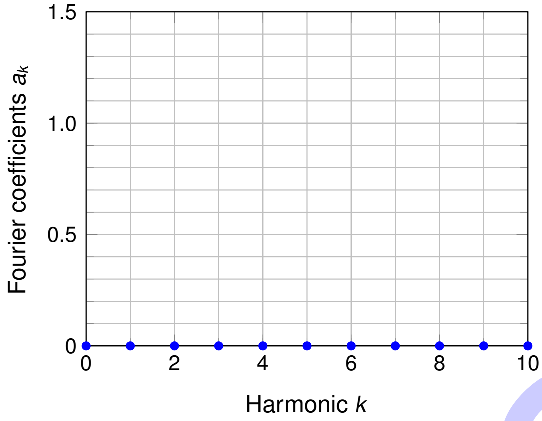
b) Given the periodic time signal $s(t)$ shown below, with $\omega = \frac{2\pi}{T}$ and $T = 1$ s. Draw the spectrum corresponding to $s(t)$ in the solution field.

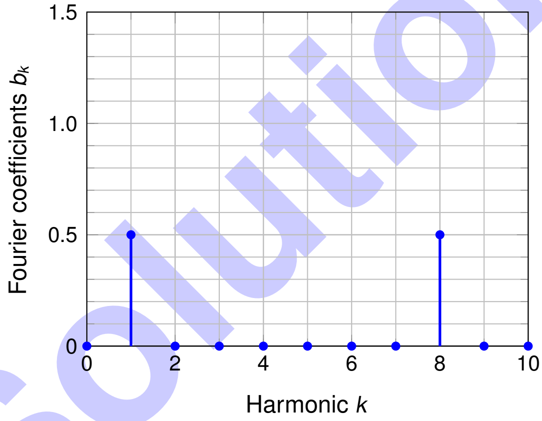




c) Given the periodic time signal $s(t)$ shown below, with $\omega = \frac{2\pi}{T}$ and $T = 1$ s. Draw the spectrum corresponding to $s(t)$ in the solution field.

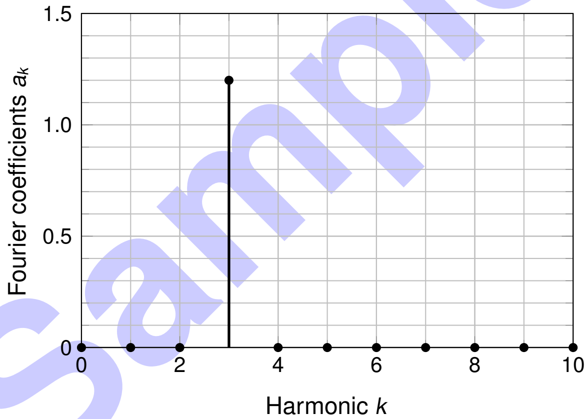


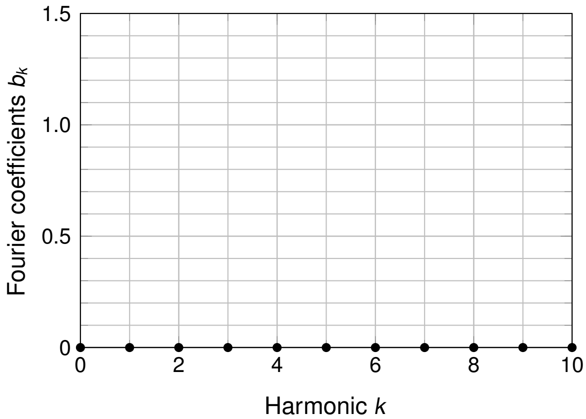


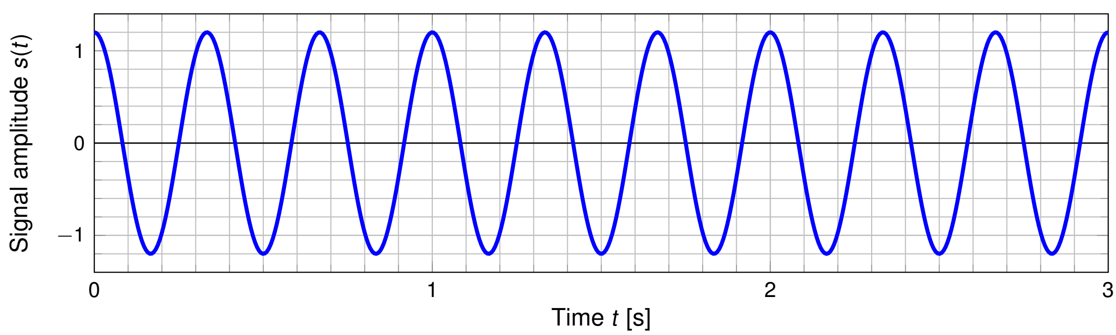


Synthesis

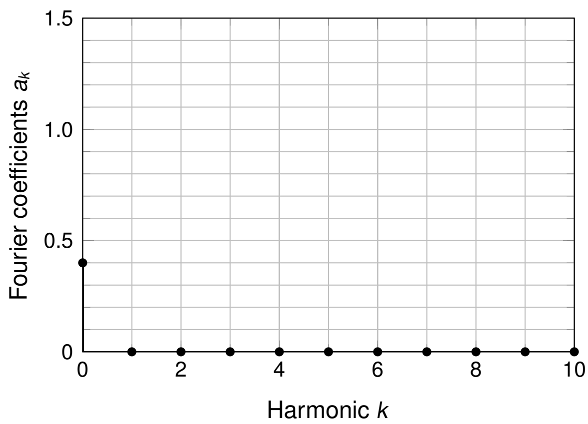
d) Given the spectrum of a Fourier Series shown below. Draw the corresponding time signal $s(t)$ in the interval $[0, 2]$ in the solution field. The constants are again $\omega = \frac{2\pi}{T}$ and $T = 1$ s.

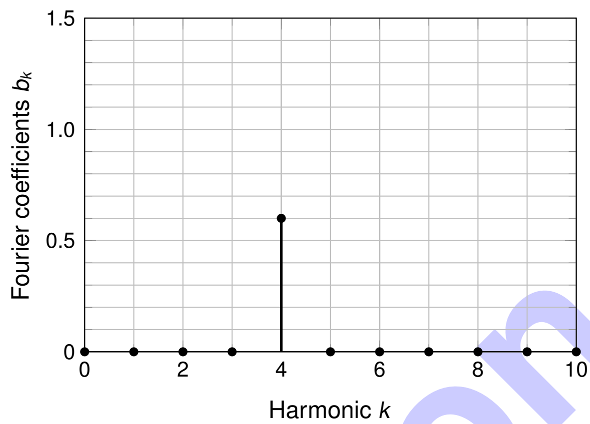


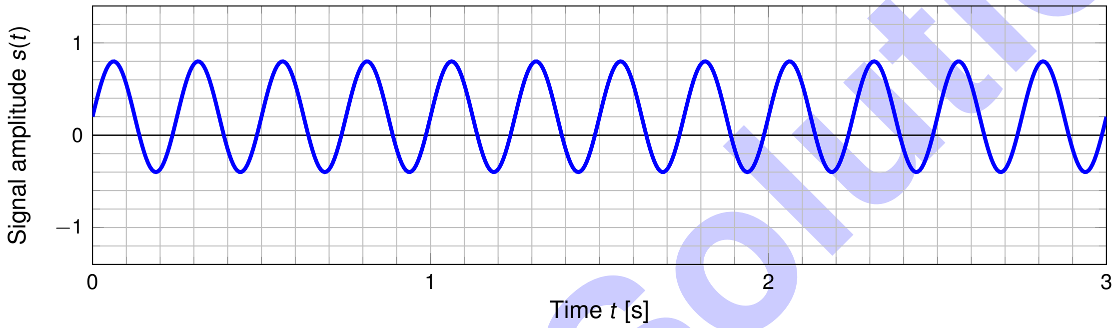




e) Given the spectrum of a Fourier Series shown below. Draw the corresponding time signal $s(t)$ in the interval $[0, 2]$ in the solution field. The constants are again $\omega = \frac{2\pi}{T}$ and $T = 1\,\mathrm{s}$.







Notes:
Exercise b) shows a single sine wave. It follows directly that all $a_k = 0$. The frequency of the sine can easily be read off. Two complete oscillations fit within the period $T = 1\,\mathrm{s}$. Therefore, $f = 2$.
In exercise c), two sine functions are now superimposed. This can be seen because the signal remains point-symmetric, which again means that all $a_k = 0$. It is apparent that the signal repeats itself every 1 s. Thus, one of the two frequencies must be $f_1 = 1$. Within this period there are eight complete sine oscillations. Therefore, $f_2 = 8$. Determining the amplitudes of the individual oscillations, i.e. b_1 and b_8, is somewhat more difficult than before and can only be estimated to some extent. To determine b_1, we now try to identify the "center" of the fundamental oscillation. If we take the midpoints of the rising and falling edges and connect them to the base oscillation, we find an amplitude of 0.5. Therefore, $b_1 = 0.5$. In the second step, if we look at how much the superimposed signal deviates from the base oscillation, we find that $b_8 = 0.5$.
In the spectrum of exercise d), $a_3 = 1.2$. This results in a cosine of frequency 3 with amplitude 1.2. All other a_k are 0. Since all $b_k = 0$ as well, there are no further (sine) oscillations.
The spectrum of exercise e) contains two peaks greater than 0. $a_0 = 2 \cdot 0.2$; therefore, the DC component equals $a_0/2 = 0.2$. Furthermore, $b_4 = 0.6$, so we obtain a sine wave with frequency 4 and amplitude 0.6, shifted upward by the DC component.

Problem 2 Quantization and channel noise

In this task, we want to quantize a temperature curve and investigate the influence of noise on signals. For this purpose, we consider temperatures in the range of –40 °C to 70 °C. The measured values are to be mapped linearly, with a step size of at most 0.5 °C.

a)* Explain the difference between sampling and quantization.

- Sampling is the discretization of a continuous signal in the time domain without rounding.
- Quantization is the discretization of a signal in signal steps, i.e. in the value range with rounding.

b)* What is the minimum number of bits required to digitize a single temperature value? Give reasons for your answer.

From the lecture we know the connection

$$M = \frac{b-a}{\Delta}, \tag{2.1}$$

where M is the number of signal steps, a and b are the lower and upper limits of the quantization interval, respectively, and Δ is the step width. Substituting $M = 220$ signal levels leads to a codeword length of $N = \lceil \log_2(M) \rceil = 8\,\text{bit}$.

c) According to subproblem b), which step size can be used to determine the temperature based on the number of bits used?

Since we have to use 8 bit to represent quantization levels anyway, in practice $M' = 256$ instead of $M = 220$ quantization levels. Solving (2.1) for Δ' and substituting yields $\Delta' \approx 0.43\,°\text{C}$

d) Determine the maximum quantization error with respect to the calculated step size from subproblem c) assuming that mathematical rounding is used.

$$\Delta'/2 = 0.43\,°\text{C} \cdot \frac{1}{2} \approx 0.215\,°\text{C}$$

If you have not solved previous subproblems, assume 256 quantization levels.[1]
The baseband signal used uses exactly one symbol for each temperature level. A channel capacity of 10 kbit/s should be achieved.

e) Determine the minimum bandwidth that is required for a noise-free channel to achieve the specified channel capacity.

Hartley's law: $C_H = 2B\log_2(M) \Rightarrow B = \frac{C_H}{2\log_2(M)} = 625\,\text{Hz}$

f) To what value would the channel capacity decrease, assuming that the same bandwidth is used and a SNR of 35 dB is applied?

Shannon's law: $C_S = B\log_2(1 + \text{SNR})$ where dB is 10 times the decadic logarithm of two equal quantities. So we get for the SNR:

$$\text{SNR} = 10\log(X) \Rightarrow X = 10^{(\text{SNR}/10)} \approx 3162.28$$

Substituting this into C_S yields:

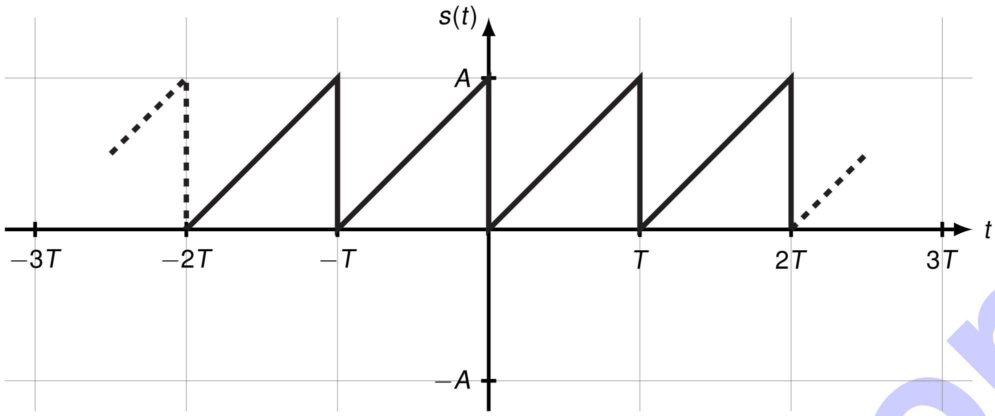
$$C_S = B\log_2(1 + X) \approx 7267\,\text{bit/s}$$

(The achievable bandwidth is always the minimum of C_H and C_S!)

[1] In the written exam, tasks basically build on each other, i. e., intermediate results of previous subproblems are to be used. For longer tasks, we — when appropriate — sometimes give substitute values so that reentry is possible.

Problem 3 Fourier Series

Given the following T-periodic time signal $s(t)$:



a)* Find an analytical expression for $s(t)$ in the interval $[0, T]$.

$$s(t) = \frac{t}{T} \cdot A \quad \text{für} \quad t \in [0, T]$$

Hint: straight line equation $y = mx + t$

The signal $s(t)$ can be developed as a Fourier series, i. h.

$$s(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega t) + b_k \sin(k\omega t)\right). \tag{3.1}$$

The coefficients a_k and b_k can be determined as follows:

$$a_k = \frac{2}{T} \int_0^T s(t) \cdot \cos(k\omega t)\, dt \quad \text{und}\ b_k = \frac{2}{T} \int_0^T s(t) \cdot \sin(k\omega t)\, dt. \tag{3.2}$$

b)* Which coefficient in formula (3.1) is responsible for the constant component of $s(t)$?

The DC component arises exclusively from a_0, because all other coefficients determine the amplitude of a sine or cosine oscillation.
Important: According to the formula 3.1, the constant component is $\frac{a_0}{2}$ (and not just a_0)!

c) Determine the constant component of the signal $s(t)$ through calculation.

Using formula (3.2) we get:

$$a_0 = \frac{2}{T} \int_0^T s(t) \cdot \cos(k\omega t)\, dt \overset{k=0}{=} \frac{2}{T} \int_0^T \frac{t}{T} \cdot A\, dt$$
$$= \frac{2}{T} \cdot \frac{A}{T} \int_0^T t\, dt = \frac{2A}{T^2} \cdot \frac{T^2}{2} = A \neq 0$$

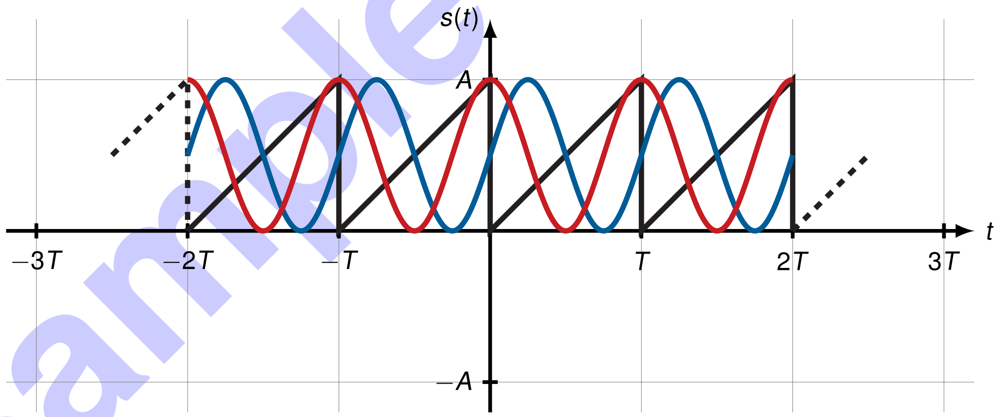
$\Rightarrow s(t)$ has a DC component. It amounts to $\frac{a_0}{2} = \frac{A}{2}$.

d)* Could you have guessed the result from the previous subtask *by inspection*?

Yes: The signal $s(t)$ only takes on values greater than zero. Therefore, it cannot be free of constant components. From the slope of the individual saw teeth it is easy to guess that the time average of the signal must be $A/2$.

e)* Determine the coefficients a_k.
Note: You do not need a calculation here. Instead, compare the symmetry of $s(t)$ with a cosine oscillation. Can a weighted cosine contribute to the overall signal?

Intuitive
The sawtooth $s(t)$ is in phase with a sine wave: To multiples of the period T, $s(t)$ has zero crossings (the DC component subtracted once). This corresponds exactly to the behaviour of a sine wave. If you do not see this, imagine the abrupt level change at multiples of T slightly slanted.
A cosine-shaped signal, on the other hand, would always have the value ± 1 at these points. However, since this does not correspond to the shape of the sawtooth, the cosine components must be removed. This is achieved by $a_k = 0, \forall k > 0$.
Mathematical
Since $\sin(x) = -\sin(-x)$ this is an odd (i.e. point-symmetrical) function. The signal $s(t)$, if one subtracts the DC component, is also point-symmetrical to the coordinate origin (otherwise the point of symmetry is merely shifted along the ordinate). The cosine, on the other hand, is a straight or axisymmetric function, which is why it cannot contribute to $s(t)$.
Through Inspection
In the figure below $s(t)$, $\cos(2\pi t)$ and $\sin(2\pi t)$ are plotted. It can be seen that the sine crosses the signal $s(t)$ exactly in its mean values for multiples of π, while the cosine takes on extreme values unequal to $s(k\pi)$ for $k \in \mathbb{Z}$.



From now on, we assume $T = 1$ for simplicity.

f)* Determine the coefficients b_k.

Hints: $\int_0^1 t \sin(ct)\, dt = \frac{\sin(c) - c \cdot \cos(c)}{c^2}$ and $\omega = 2\pi/T$.

The hint saves us from partial integration. We obtain:

$$b_k = \frac{2}{T} \int_0^T s(t) \sin(k\omega t)\, dt = 2A \int_0^1 t \sin(k\omega t)\, dt \tag{3.3}$$

$$= 2A \cdot \frac{\sin(k\omega) - k\omega \cos(k\omega)}{k^2\omega^2} \overset{\omega=2\pi}{=} -\frac{A}{k\pi} \tag{3.4}$$

g) **Homework:** Using the results so far, sketch the DC component $a_0/2$, the first two harmonics and their sum for $A = \pi$ in a coordinate system.

For $A = \pi$ we get:

$$\frac{a_0}{2} = \frac{\pi}{2} \approx 1.6, \; b_1 = -1, \; b_2 = -\frac{1}{2}.$$

The first two harmonics are as follows

$$h_1(t) = b_1 \sin(2\pi t) = -\sin(2\pi t), \quad \text{und} \quad h_2(t) = b_2 \sin(4\pi t) = -\frac{1}{2}\sin(4\pi t).$$